

ON ITERATION OF COX RINGS

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ABSTRACT. We characterize all varieties with a torus action of complexity one that admit iteration of Cox rings.

1. INTRODUCTION

We consider normal algebraic varieties X defined over the field \mathbb{C} of complex numbers. If X has finitely generated divisor class group K and only constant invertible global regular functions, then one defines the K -graded Cox ring R_1 of X as follows, see [2] for details:

$$R_1 = \bigoplus_K \Gamma(X, \mathcal{O}(D)).$$

If the Cox ring R_1 is a finitely generated \mathbb{C} -algebra, then one has the total coordinate space $X_1 := \operatorname{Spec} R_1$. We say that X admits *iteration of Cox rings* if there is a chain

$$X_p \xrightarrow{\parallel H_{p-1}} X_{p-1} \xrightarrow{\parallel H_{p-2}} \dots \xrightarrow{\parallel H_2} X_2 \xrightarrow{\parallel H_1} X_1$$

dominated by a factorial variety X_p where in each step, X_{i+1} is the total coordinate space of X_i and $H_i = \operatorname{Spec} \mathbb{C}[K_i]$ the characteristic quasitorus of X_i , having the divisor class group K_i of X_i as its character group. Note that if the divisor class group K of X is torsion free, then R_1 is a unique factorization domain and iteration of Cox rings is trivially possible. As soon as K has torsion, it may happen that during the iteration process a total coordinate space with non-finitely generated divisor class group pops up and thus there is no chain of total coordinate spaces as above, see [1, Rem. 5.12].

In [1] we studied normal, rational, \mathbb{T} -varieties X of complexity one, where the latter means that X comes with an effective torus action $\mathbb{T} \times X \rightarrow X$ such that $\dim(\mathbb{T}) = \dim(X) - 1$ holds. We showed that for affine X with $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}$ and at most log terminal singularities, the iteration of Cox rings is possible. In the present article, we characterize all varieties X with a torus action of complexity one that admit iteration of Cox rings.

First consider the case $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}$. In order to have finitely generated divisor class group, X must be rational and then the Cox ring of X is of the form $R = \mathbb{C}[T_{ij}, S_k]/I$, with a polynomial ring $\mathbb{C}[T_{ij}, S_k]$ in variables T_{ij} and S_k modulo the ideal I generated by the trinomial relations

$$T_0^{l_0} + T_1^{l_1} + T_2^{l_2}, \quad \theta_1 T_1^{l_1} + T_2^{l_2} + T_3^{l_3}, \quad \dots, \quad \theta_{r-2} T_{r-2}^{l_{r-2}} + T_{r-1}^{l_{r-1}} + T_r^{l_r},$$

with $T_i^{l_i} = T_{i1}^{l_{i1}} \dots T_{in_i}^{l_{in_i}}$. For each exponent vector l_i set $l_i := \gcd(l_{i1}, \dots, l_{in_i})$. We say that R is *hyperplatonic* if $l_0^{-1} + \dots + l_r^{-1} > r - 1$ holds. After reordering l_0, \dots, l_r decreasingly, the latter condition precisely means that $l_i = 1$ holds for all $i \geq 3$ and (l_0, l_1, l_2) is a platonic triple, i.e., a triple of the form

$$(5, 3, 2), \quad (4, 3, 2), \quad (3, 3, 2), \quad (x, 2, 2), \quad (x, y, 1), \quad x, y \in \mathbb{Z}_{\geq 1}.$$

Theorem 1.1. *Let X be a normal \mathbb{T} -variety of complexity one with $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}$. Then the following statements are equivalent.*

- (i) *The variety X admits iteration of Cox rings.*
- (ii) *The variety X is rational with hyperplatonic Cox ring.*

We turn to the case $\Gamma(X, \mathcal{O})^{\mathbb{T}} \neq \mathbb{C}$. Here, $\mathcal{O}(X)^* = \mathbb{C}^*$ and finite generation of the divisor class group of X force $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}[T]$. In this situation, we obtain the following simple characterization.

Theorem 1.2. *Let X be a normal \mathbb{T} -variety of complexity one with $\Gamma(X, \mathcal{O})^{\mathbb{T}} \neq \mathbb{C}$. Then X admits Cox ring iteration if and only if X and its total coordinate space are rational. Moreover, if the latter holds, then the Cox ring iteration stops after at most one step.*

As a consequence of the two theorems above, we obtain the following structural result, generalizing [1, Thm. 3], but using analogous ideas for the proof.

Corollary 1.3. *Let X be a normal, rational variety with a torus action of complexity one admitting iteration of Cox rings. Then X is a quotient $X = X' // G$ of a factorial affine variety $X' := \text{Spec}(R')$, where R' is a factorial ring and G is a solvable reductive group.*

On our way of proving Theorem 1.1, we give in Proposition 2.6 an explicit description of the Cox ring of a variety $\text{Spec } R$ for a hyperplatonic ring R . This allows to describe the possible Cox ring iteration chains more in detail. After reordering the numbers l_0, \dots, l_r associated with R decreasingly, we call (l_0, l_1, l_2) the *basic platonic triple* of R .

Corollary 1.4. *The possible sequences of basic platonic triples arising from Cox ring iterations of normal, rational varieties with a torus action of complexity one and hyperplatonic Cox ring are the following:*

- (i) $(1, 1, 1) \rightarrow (2, 2, 2) \rightarrow (3, 3, 2) \rightarrow (4, 3, 2)$,
- (ii) $(1, 1, 1) \rightarrow (x, x, 1) \rightarrow (2x, 2, 2)$,
- (iii) $(1, 1, 1) \rightarrow (x, x, 1) \rightarrow (x, 2, 2)$,
- (iv) $(l_{01}^{-1}l_0, l_{01}^{-1}l_1, 1) \rightarrow (l_0, l_1, 1)$, where $l_{01} := \gcd(l_0, l_1) > 1$.

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2. PROOF OF THEOREM 1.1

We will work in the notation of [3, 5], where the Cox ring of a rational T -variety of complexity one is encoded by a pair of defining matrices. Let us briefly recall the precise definitions we need from [5]; note that the setting will be slightly more flexible than the informal one given in the introduction.

Construction 2.1. Fix integers $r, n > 0$, $m \geq 0$ and a partition $n = n_0 + \dots + n_r$. For every $i = 0, \dots, r$, fix a tuple $l_i \in \mathbb{Z}_{>0}^{n_i}$ and define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \in \mathbb{C}[T_{ij}, S_k; 0 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m].$$

We will also write $\mathbb{C}[T_{ij}, S_k]$ for the above polynomial ring. Let $A := (a_0, \dots, a_r)$ be a $2 \times (r+1)$ matrix with pairwise linearly independent columns $a_i \in \mathbb{C}^2$. For every $i = 0, \dots, r-2$ we define

$$g_i := \det \begin{bmatrix} T_i^{l_i} & T_{i+1}^{l_{i+1}} & T_{i+2}^{l_{i+2}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix} \in \mathbb{C}[T_{ij}, S_k].$$

We build up an $r \times (n+m)$ matrix from the exponent vectors l_0, \dots, l_r of these polynomials:

$$P_0 := \begin{bmatrix} -l_0 & l_1 & & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -l_0 & 0 & & l_r & 0 & \dots & 0 \end{bmatrix}.$$

Denote by P_0^* the transpose of P_0 and consider the projection

$$Q: \mathbb{Z}^{n+m} \rightarrow K_0 := \mathbb{Z}^{n+m} / \text{im}(P_0^*).$$

Denote by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables T_{ij}, S_k . Define a K_0 -grading on $\mathbb{C}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) := Q(e_{ij}) \in K_0, \quad \deg(S_k) := Q(e_k) \in K_0.$$

This is the finest possible grading of $\mathbb{C}[T_{ij}, S_k]$ leaving the variables and the g_i homogeneous. In particular, we have a K_0 -graded factor algebra

$$R(A, P_0) := \mathbb{C}[T_{ij}, S_k] / \langle g_0, \dots, g_{r-2} \rangle.$$

By the results of [3, 5] the rings $R(A, P_0)$ are normal complete intersections, admit only constant homogeneous units and we have unique factorization in the multiplicative monoid of K_0 -homogeneous elements of $R(A, P_0)$. Moreover, suitably downgrading the rings $R(A, P_0)$ leads to the Cox rings of the normal rational T -varieties X of complexity one with $\Gamma(X, \mathcal{O})^\mathbb{T} = \mathbb{C}$, see [4, 3, 5].

In order to iterate a Cox ring $R(A, P_0)$, it is necessary that $\text{Spec } R(A, P_0)$ has finitely generated divisor class group. The latter turns out to be equivalent to rationality of $\text{Spec } R(A, P_0)$. From [1, Cor. 5.8], we infer the following rationality criterion.

Remark 2.2. Let $R(A, P_0)$ be as in Construction 2.1 and set $\mathfrak{l}_i := \gcd(l_{i1}, \dots, l_{in_i})$. Then $\text{Spec } R(A, P_0)$ is rational if and only if one of the following conditions holds:

- (i) We have $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $0 \leq i < j \leq r$, in other words, $R(A, P_0)$ is factorial.
- (ii) There are $0 \leq i < j \leq r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) > 1$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j\}$.
- (iii) There are $0 \leq i < j < k \leq r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = \gcd(\mathfrak{l}_i, \mathfrak{l}_k) = \gcd(\mathfrak{l}_j, \mathfrak{l}_k) = 2$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j, k\}$.

Definition 2.3. Let $R(A, P_0)$ be as in Construction 2.1 such that $\text{Spec } R(A, P_0)$ is rational. We say that P_0 is *gcd-ordered* if it satisfies the following two properties

- (i) $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $i = 0, \dots, r$ and $j = 3, \dots, r$,
- (ii) $\gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$.

Observe that if $\text{Spec } R(A, P_0)$ is rational, then one can always achieve that P_0 is gcd-ordered by suitably reordering l_0, \dots, l_r . This does not affect the K_0 -graded algebra $R(A, P_0)$ up to isomorphism.

Lemma 2.4. Let $R(A, P_0)$ be as in Construction 2.1 such that $\text{Spec } R(A, P_0)$ is rational and P_0 is gcd-ordered. Then, with $K_0 = \mathbb{Z}^{n+m} / \text{im}(P_0^*)$, the kernel of

$\mathbb{Z}^{n+m} \rightarrow K_0/K_0^{\text{tors}}$ is generated by the rows of

$$P_1 := \begin{bmatrix} \frac{-1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)} \mathfrak{l}_0 & \frac{1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)} \mathfrak{l}_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \frac{-1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_2)} \mathfrak{l}_0 & 0 & \frac{1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_2)} \mathfrak{l}_2 & 0 & 0 & & & \\ -\mathfrak{l}_0 & 0 & & \mathfrak{l}_3 & 0 & \vdots & & \vdots \\ \vdots & & & \vdots & \ddots & \vdots & & \\ -\mathfrak{l}_0 & 0 & \dots & 0 & & \mathfrak{l}_r & 0 & \dots & 0 \end{bmatrix}.$$

Proof. The arguments are similar as for [1, Cor. 6.3]. The row lattice of P_0 is a sublattice of finite index of that of P_1 and thus there is a commutative diagram

$$\begin{array}{ccc} K_0 & \xrightarrow{\quad} & K_0/K_0^{\text{tors}} \\ & \searrow & \nearrow \\ & \mathbb{Z}^{n+m}/\text{im}(P_1^*) & \end{array}$$

We have to show, that $\mathbb{Z}^{n+m}/\text{im}(P_1^*)$ is torsion free. Suitable elementary column operations on P_1 reduce the problem to showing that for the $r \times (r+1)$ matrix

$$\begin{bmatrix} \frac{-1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)} \mathfrak{l}_0 & \frac{1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)} \mathfrak{l}_1 & 0 & \dots & 0 \\ \frac{-1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_2)} \mathfrak{l}_0 & 0 & \frac{1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_2)} \mathfrak{l}_2 & 0 & 0 \\ -\mathfrak{l}_0 & 0 & & \mathfrak{l}_3 & 0 \\ \vdots & & & \vdots & \ddots & \vdots \\ -\mathfrak{l}_0 & 0 & \dots & 0 & & \mathfrak{l}_r \end{bmatrix}$$

the r -th determinantal divisor and therefore the product of the invariant factors equals one. Up to sign, the $r \times r$ minors of the above matrix are

$$\frac{1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1) \gcd(\mathfrak{l}_0, \mathfrak{l}_2)} \mathfrak{l}_0 \cdots \mathfrak{l}_{i-1} \cdot \mathfrak{l}_{i+1} \cdots \mathfrak{l}_r, \quad \text{where } i = 0, \dots, r.$$

Suppose that some prime p divides all these minors. Then $p \nmid \mathfrak{l}_j$ holds for all $j \geq 3$, because otherwise we find an $i \neq j$ with $p \mid \mathfrak{l}_i$, contradicting gcd-orderedness of P_0 . Thus, p divides each of the numbers

$$\frac{\mathfrak{l}_0 \mathfrak{l}_2}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1) \gcd(\mathfrak{l}_0, \mathfrak{l}_2)}, \quad \frac{\mathfrak{l}_1 \mathfrak{l}_2}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1) \gcd(\mathfrak{l}_0, \mathfrak{l}_2)}, \quad \frac{\mathfrak{l}_0 \mathfrak{l}_1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1) \gcd(\mathfrak{l}_0, \mathfrak{l}_2)}.$$

By the assumption of the lemma, $\mathfrak{l} := \gcd(\mathfrak{l}_1, \mathfrak{l}_2)$ equals $\gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$. Consequently, we obtain

$$\gcd(\mathfrak{l}_0 \mathfrak{l}_2, \mathfrak{l}_0 \mathfrak{l}_1, \mathfrak{l}_1 \mathfrak{l}_2) = \gcd(\mathfrak{l}_0 \mathfrak{l}, \mathfrak{l}_1 \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_1) \gcd(\mathfrak{l}_0, \mathfrak{l}_2).$$

We conclude $p = 1$; a contradiction. Being the greatest common divisor of the above minors, the r -th determinantal divisor equals one. \square

Lemma 2.5. *Let $R(A, P_0)$ be as in Construction 2.1 and $X := \text{Spec } R(A, P_0)$ be rational. Assume that P_0 is gcd-ordered. Then the number $c(i)$ of irreducible components of $V(X, T_{ij})$ is given by*

i	0	1	2	≥ 3
$c(i)$	$\gcd(\mathfrak{l}_1, \mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0, \mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0, \mathfrak{l}_1)$	$\frac{1}{\mathfrak{l}} \gcd(\mathfrak{l}_1, \mathfrak{l}_2) \gcd(\mathfrak{l}_0, \mathfrak{l}_2) \gcd(\mathfrak{l}_0, \mathfrak{l}_1)$

Proof. The assertion is a direct consequence of [1, Lemma 6.4]. \square

We are ready for the main ingredience of the proof of Theorem 1.1, the explicit description of the iterated Cox ring.

Proposition 2.6. *Let $R(A, P_0)$ be non-factorial with $\text{Spec } R(A, P_0)$ rational. Assume that P_0 is gcd-ordered and let P_1 be as in Lemma 2.4. Define numbers $n' := c(0)n_0 + \dots + c(r)n_r$ and*

$$n_{i,1}, \dots, n_{i,c(i)} := n_i, \quad l_{ij,1}, \dots, l_{ij,c(i)} := \gcd((P_1)_{1,ij}, \dots, (P_1)_{r,ij}).$$

Then the vectors $l_{i,\alpha} := (l_{i1,\alpha}, \dots, l_{in_i,\alpha}) \in \mathbb{Z}^{n_i,\alpha}$ build up an $r' \times (n' + m)$ matrix P'_0 . With a suitable matrix A' , the affine variety $\text{Spec } R(A', P'_0)$ is the total coordinate space of the affine variety $\text{Spec } R(A, P_0)$.

Proof. The idea is to work with the action of the torus $H_0^0 := \text{Spec } \mathbb{C}[K_0/K_0^{\text{tors}}]$ on $X := \text{Spec } R(A, P_0)$ and to use the description of the Cox ring of a variety with torus action provided in [4]. For this, one has to look at the exceptional fibers of the map $\pi: X_0 \rightarrow Y$, where $X_0 \subseteq X$ is the set of points with at most finite H_0^0 -isotropy and the curve Y is the separation of X_0/H_0^0 . Following the lines of the proof of [1, Prop. 6.6], one uses Lemma 2.5 to determine the number of components for each fiber of π and Lemma 2.4 to determine the order of the general (finite) H_0^0 -isotropy groups on each component. The rest is application of [4]. \square

If $R(A, P_0)$ is a hyperplatonic ring, then $l_0^{-1} + \dots + l_r^{-1} \geq r - 1$ holds. Thus, we find a (unique) platonic triple (l_i, l_j, l_k) with i, j, k pairwise different and all l_u with u different from i, j, k equal one. We call (l_i, l_k, l_k) the *basic platonic triple* (bpt) of $R(A, P_0)$.

Remark 2.7. Let $R(A, P_0)$ be non-factorial and hyperplatonic with basic platonic triple (l_0, l_1, l_2) . Then Remark 2.2 ensures that $X := \text{Spec } R(A, P_0)$ is rational. Moreover, Lemma 2.5 and Proposition 2.6 yield that the exponent vectors of the defining relations of the Cox ring $R(A', P'_0)$ of X are computed in terms of the exponent vectors l_0, \dots, l_r of $R(A, P_0)$ according to the table below, where “ $a \times l_i$ ” means that the vector l_i shows up a times:

bpt of $R(A, P_0)$	exponent vectors in $R(A', P'_0)$
$(4, 3, 2)$	$2 \times l_1, \frac{1}{2}l_0, \frac{1}{2}l_2$ and $2 \times l_i$ for $i \geq 3$
$(3, 3, 2)$	$3 \times l_2, \frac{1}{3}l_0, \frac{1}{3}l_1$ and $3 \times l_i$ for $i \geq 3$
$(x, 2, 2)$ and $2 \mid x$	$2 \times \frac{1}{2}l_0, 2 \times \frac{1}{2}l_1, 2 \times \frac{1}{2}l_2$ and $4 \times l_i$ for $i \geq 3$
$(x, 2, 2)$ and $2 \nmid x$	$2 \times l_0, \frac{1}{2}l_1, \frac{1}{2}l_2$ and $2 \times l_i$ for $i \geq 3$
$(x, y, 1)$	$\frac{1}{\gcd(x,y)}l_0, \frac{1}{\gcd(x,y)}l_1$ and $\gcd(x, y) \times l_i$ for $i \geq 2$

Lemma 2.8. *Let $R(A, P_0)$, arising from Construction 2.1, be non-factorial and assume that $X := \text{Spec } R(A, P_0)$ is rational. If the total coordinate space of X is rational as well, then $l_i > 1$ holds for at most three $0 \leq i \leq r$.*

Proof. We may assume that P_0 is gcd-ordered. Then Proposition 2.6 provides us with the exponent vectors of the Cox ring $R(A', P'_0)$ of X . As $R(A, P_0)$ is rational and non-factorial, Remark 2.2 leaves us with the following two cases.

Case 1. We have $\gcd(l_0, l_1) > 1$ and $\gcd(l_i, l_j) = 1$ whenever $j \geq 2$. This means in particular $l_0, l_1 > 1$. Assume that there are $2 \leq i < j \leq r$ with $l_i, l_j > 1$. According to Proposition 2.6, we find $c(i)$ times the exponent vector l_i and $c(j)$ times the exponent vector l_j in P'_0 . Lemma 2.5 tells us $c(j) = c(i) = \gcd(l_0, l_1) > 1$. Thus, for the first two copies of l_i and l_j , we obtain $\gcd(l_{i,1}, l_{j,2}) = l_i > 1$ and $\gcd(l_{j,1}, l_{i,2}) = l_j > 1$ respectively. Remark 2.2 shows that $\text{Spec } R(A', P'_0)$ is not rational; a contradiction.

Case 2. We have $\gcd(l_0, l_1) = \gcd(l_0, l_2) = \gcd(l_1, l_2) = 2$. Assume that there is an index $3 \leq i \leq r$ with $l_i > 1$. Proposition 2.6 and Lemma 2.5 yield that the exponent vector l_i occurs $c(k) = 4$ times in the matrix P'_0 . As in the previous case we conclude via Remark 2.2 that the total coordinate space $\text{Spec } R(A', P'_0)$ is not rational; a contradiction. \square

Proof of Theorem 1.1. We prove “(ii) \Rightarrow (i)”. Then X is a rational and has a hyperplatonic ring $R(A, P_0)$ provided by Construction 2.1 as its Cox ring. If $R(A, P_0)$ is factorial, then there is nothing to show. So, let $R(A, P_0)$ be non-factorial. We may assume that P_0 is gcd-ordered. Then (l_0, l_1, l_2) is the basic platonic triple of $R(A, P_0)$. From Remark 2.7 we infer that $X_1 := \text{Spec } R(A, P_0)$ is rational with hyperplatonic Cox ring $R(A', P'_0)$. So, we can pass to $X_2 := R(A', P'_0)$ and so forth. The table of possible basic platonic triples given in Remark 2.7 shows that the iteration process terminates at a factorial ring.

We prove “(i) \Rightarrow (ii)”. Since X has a Cox ring, X must have finitely generated divisor class group. As for any \mathbb{T} -variety of complexity one, the latter is equivalent to X being rational. The Cox ring of X is a ring $R(A, P_0)$ as provided by Construction 2.1. If $R(A, P_0)$ is factorial, then we are done. So, let $R(A, P_0)$ be non-factorial. Then we may assume that P_0 is gcd-ordered and, moreover, $l_{01} \neq 1$. Since $X_1 = \text{Spec } R(A, P_0)$ has a Cox ring $R(A', P'_0)$, it must be rational. By Lemma 2.8 we have $l_j = 1$ whenever $j \geq 3$ holds. Remark 2.2 leaves us with the following cases.

Case 1. We have $l_{01} := \gcd(l_0, l_1) > 1$ and $\gcd(l_i, l_j) = 1$ whenever $j \geq 2$ holds. Then we may assume $l_0 \geq l_1$.

1.1. Consider the case $l_{01} > 3$. By Lemma 2.5, the exponent vector l_2 occurs l_{01} times in the defining relations of the Cox ring $R(A', P'_0)$ of X_1 . Since $\text{Spec } R(A', P'_0)$ is rational, Remark 2.2 yields $l_2 = 1$. We conclude that (l_0, l_1, l_2) is platonic.

1.2. Assume $l_{01} = 3$. Then l_2 occurs 3 times as exponent vector in the defining relations of $R(A', P'_0)$. Remark 2.2 shows $l_2 \leq 2$. Thus, (l_0, l_1, l_2) is platonic.

1.3. Let $l_{01} = 2$. If $l_0 = l_1 = 2$ holds, then (l_0, l_1, l_2) is a platonic triple for any l_2 . So, assume $l_0 > l_1 \geq 2$. As we are in Case 1, the number l_2 must be odd. If $l_2 = 1$ holds, then (l_0, l_1, l_2) is a platonic triple. By Proposition 2.6 and Lemma 2.5, we find the exponent vectors $1/2 l_0$ and $1/2 l_1$ as well as twice l_2 in P'_0 . Since $X_1 = \text{Spec } R(A', P'_0)$ is rational and $l_0 > l_1$ holds, Lemma 2.8 shows $l_1 = 2$ and the triple of non-trivial gcd's of exponent vectors of P'_0 is $(l_0/2, l_2, l_2)$. After gcd-ordering P'_0 , we can apply Case 1.1 and with $l_0/2 > 1$ we obtain $l_0 = 4$ and $l_2 = 3$. In particular, (l_0, l_1, l_2) is platonic.

Case 2: We have $\gcd(l_0, l_1) = \gcd(l_0, l_2) = \gcd(l_1, l_2) = 2$. Then we may assume $l_0 \geq l_1 \geq l_2$. Proposition 2.6 and Lemma 2.5 tell us that each of the exponent vectors $1/2 l_0$, $1/2 l_1$ and $1/2 l_2$ occurs twice in P'_0 . Since $\text{Spec } R(A', P'_0)$ is rational, Lemma 2.8 yields $l_1 = l_2 = 2$. Thus, (l_0, l_1, l_2) is platonic. \square

3. PROOF OF THEOREM 1.2

As a first step we relate the total coordinate space of a rational variety with torus action of complexity one admitting non-constant invariant functions to the total coordinate space of one with only constant invariant functions; see Corollary 3.4. This allows us to characterize rationality of the total coordinate space using previous results; see Corollary 3.5. Then we determine in a similar manner as before, the iterated Cox ring; see Proposition 3.7. This finally allows us to prove Theorem 1.2. We begin with recalling the necessary notions from [5].

Construction 3.1. Fix integers $r, n > 0$, $m \geq 0$ and a partition $n = n_1 + \dots + n_r$. For each $1 \leq i \leq r$, fix a tuple $l_i \in \mathbb{Z}_{>0}^{n_i}$ and define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \dots T_{in_i}^{l_{in_i}} \in \mathbb{C}[T_{ij}, S_k; 1 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m].$$

Let $A := (a_1, \dots, a_r)$ be a list of pairwise different elements of \mathbb{C} . Define for every $i = 1, \dots, r-1$ a polynomial

$$g_i := T_i^{l_i} - T_{i+1}^{l_{i+1}} - (a_{i+1} - a_i) \in \mathbb{C}[T_{ij}, S_k].$$

We build up an $r \times (n + m)$ matrix from the exponent vectors l_1, \dots, l_r of these polynomials:

$$P_0 := \begin{bmatrix} l_1 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & l_r & 0 & \dots & 0 \end{bmatrix}.$$

Similar to the case in Construction 2.1 the matrix P_0 defines a grading of the group $K_0 := \mathbb{Z}^{n+m}/\text{im}(P_0^*)$ on the ring

$$R(A, P_0) := \mathbb{C}[T_{ij}, S_k] / \langle g_1, \dots, g_{r-1} \rangle.$$

Following [5] we call a ring $R(A, P_0)$ arising from Construction 3.1 of *Type 1* and a ring $R(A, P_0)$ as in Construction 2.1 of *Type 2*. According to [5], the suitable downgradings of the rings $R(A, P_0)$ of Type 1 yield precisely the Cox rings of the normal rational \mathbb{T} -varieties X of complexity one with $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}[T]$.

Construction 3.2. Consider a ring $R(A, P_0)$ of Type 1. Set $l_i := \gcd(l_{i1}, \dots, l_{in_i})$ and $\ell := \text{lcm}(l_1, \dots, l_r)$. Then, writing L_0 for the column vector $-(\ell, \dots, \ell) \in \mathbb{Z}^r$, we obtain a ring $R(\tilde{A}, \tilde{P}_0)$ of Type 2 with defining matrices

$$\tilde{A} := \begin{bmatrix} -1 & a_1 & \dots & a_r \\ 0 & 1 & \dots & 1 \end{bmatrix}, \quad \tilde{P}_0 := [L_0, P_0].$$

Proposition 3.3. Let $R(A, P_0)$ be a ring of Type 1 and $R(\tilde{A}, \tilde{P}_0)$ the associated ring of Type 2 obtained via Construction 3.2. Fix $\alpha_{ij} \in \mathbb{Z}$ with $l_i = \alpha_{i1}l_{i1} + \dots + \alpha_{in_i}l_{in_i}$. Then one obtains an isomorphism of graded \mathbb{C} -algebras

$$R(\tilde{A}, \tilde{P}_0)_{\tilde{T}_{01}} \rightarrow R(A, P_0)[T_{01}, T_{01}]^{-1}, \quad \tilde{T}_{01} \mapsto T_{01}, \quad \tilde{T}_{ij} \mapsto T_{ij} T_{01}^{\frac{\ell}{l_i} \alpha_{ij}}.$$

Proof. By construction, $R(\tilde{A}, \tilde{P}_0)$ is a factor algebra of $\mathbb{C}[\tilde{T}_{ij}, \tilde{S}_k]$ and $R(A, P_0)$ of $\mathbb{C}[T_{ij}, S_k]$. We have an isomorphism of \mathbb{C} -algebras

$$\psi: \mathbb{C}[\tilde{T}_{ij}, \tilde{S}_k]_{\tilde{T}_{01}} \rightarrow \mathbb{C}[T_{ij}, S_k][T_{01}, T_{01}]^{-1}, \quad \tilde{T}_{01} \mapsto T_{01}, \quad \tilde{T}_{ij} \mapsto T_{ij} T_{01}^{\frac{\ell}{l_i} \alpha_{ij}}, \quad \tilde{S}_k \mapsto S_k.$$

Observe $\psi(\tilde{T}_i^{l_i}) = T_{01}^{\ell} T_i^{l_i}$. We claim that ψ is compatible with the gradings by \tilde{K}_0 on the l.h.s. and by $\mathbb{Z} \times K_0$ on the r.h.s., where the latter grading is given by

$$\deg(T_{01}) = (1, 0) \in \mathbb{Z} \times K_0, \quad \deg(T_{ij}) = (0, e_{ij} + \text{im}(P_0^*)) \in \mathbb{Z} \times K_0.$$

Indeed, because of $\psi(\tilde{T}_{01}^{-\ell} \tilde{T}_i^{l_i}) = T_i^{l_i}$, the kernels of the respective downgrading maps

$$\mathbb{Z}^{n+1+m} \rightarrow \tilde{K}_0, \quad \mathbb{Z}^{n+1+m} \rightarrow \mathbb{Z} \times K_0,$$

generated by the rows \tilde{P}_0 and P_0 , correspond to each other under ψ . The defining ideal of $R(\tilde{A}, \tilde{P}_0)$ is generated by the polynomials $\tilde{g}_1, \dots, \tilde{g}_{r-1}$, where

$$\tilde{g}_i := \det \begin{bmatrix} \tilde{T}_0^{\ell} & T_i^{l_i} & T_{i+1}^{l_{i+1}} \\ -1 & a_i & a_{i+1} \\ 0 & 1 & 1 \end{bmatrix}.$$

The above isomorphism sends \tilde{g}_i to $T_{01}^{\ell} g_i$, where the g_i are the generators of the defining ideal of $R(A, P_0)$, and thus induces the desired isomorphism. \square

Corollary 3.4. *Let $X := \operatorname{Spec} R(A, P_0)$ be the affine variety arising from a ring of Type 1 and $\tilde{X} := \operatorname{Spec} R(\tilde{A}, \tilde{P}_0)$ the one arising from the associated ring of Type 2. Then $X \times \mathbb{C}^*$ is isomorphic to the principal open subset $\tilde{X}_{\tilde{T}_{01}} \subseteq \tilde{X}$. In particular, X is rational if and only if \tilde{X} is so.*

Corollary 3.5. *Let $R(A, P_0)$ be a ring of Type 1. Then $X = \operatorname{Spec} R(A, P_0)$ is rational if and only if one of the following conditions holds:*

- (i) *One has $l_i = 1$ for all $1 \leq i \leq r$, in other words, $R(A, P_0)$ is factorial.*
- (ii) *There is exactly one $1 \leq i \leq r$ with $l_i > 1$.*
- (iii) *There are $1 \leq i < j \leq r$ with $l_i = l_j = 2$ and $l_u = 1$ whenever $u \notin \{i, j\}$*

Proof. Combine Corollary 3.4 with the rationality criterion Remark 2.2. \square

Lemma 3.6. *Let $R(A, P_0)$ be of Type 1 with $X := \operatorname{Spec} R(A, P_0)$ rational and assume that (l_1, \dots, l_r) is decreasingly ordered. Then the number $c(i)$ of irreducible components of $V(X, T_{ij})$ is given as*

$$\begin{array}{c|c|c|c} i & 1 & 2 & \geq 3 \\ \hline c(i) & l_1 & l_2 & l_1 l_2 \end{array}$$

Proof. Due to Corollary 3.4, we can realize $X \times \mathbb{C}^*$ as a principal open subset of the associated variety \tilde{X} of Type 2. Then the irreducible components of $V(X, T_{ij}) \times \mathbb{C}^*$ are in one-to-one correspondence with the irreducible components $X \cap V(\tilde{X}, \tilde{T}_{ij})$. The assertions follows. \square

Proposition 3.7. *Let $R(A, P_0)$ be non-factorial of Type 1 with $\operatorname{Spec} R(A, P_0)$ rational and (l_1, \dots, l_r) decreasingly ordered. Define numbers $n' := c(1)n_1 + \dots + c(r)n_r$ and*

$$n_{i,1}, \dots, n_{i,c(i)} := n_i, \quad l_{i,1}, \dots, l_{i,c(i)} := \frac{1}{l_i} l_i.$$

Then the vectors $l_{i,\alpha} \in \mathbb{Z}^{n_i, \alpha}$ build up an $r' \times (n' + m)$ matrix P'_0 . With a suitable matrix A' the affine variety $\operatorname{Spec} R(A', P'_0)$ is the total coordinate space of the affine variety $\operatorname{Spec} R(A, P_0)$.

Proof. First observe that the kernel of $\mathbb{Z}^{n+m} \rightarrow K_0/K_0^{\text{tors}}$ is generated by the rows of the following $r \times (n + m)$ matrix:

$$\begin{bmatrix} \frac{1}{l_1} l_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & \frac{1}{l_r} l_r & 0 & \dots & 0 \end{bmatrix}.$$

Now one determines the Cox ring of $X = \operatorname{Spec} R(A, P_0)$ in the same manner as in the proof of [1, Prop. 6.6] by exchanging the matrix P_1 used there by the matrix above and applying Lemma 3.6. \square

Proof of Theorem 1.2. If $R(A, P_0)$ is rational of Type 1, then Proposition 3.7 shows that the Cox ring of $\operatorname{Spec} R(A, P_0)$ is factorial. Thus, Cox ring iteration is possible for X if and only if the total coordinate space of X is rational. Moreover, if the latter holds then the Cox ring iteration ends with at most one step. \square

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